ELG HS.F.4: Build a function that models a relationship between two quantities

Vertical Progression:

7th Grade	**7.RP.A Analyze proportional relationships and use them to solve real-world and mathematical problems.** ○ **7.RP.A.2** Recognize and represent proportional relationships between quantities. ○ **7.RP.A.2a.** Decide whether two quantities are in a proportional relationship. ○ **7.RP.A.2b.** Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships. ○ **7.RP.A.2c.** Represent proportional relationships by equations. ○ **7.RP.A.2d.** Explain what a point (*x, y*) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, *r*) where *r* is the unit rate.
8th Grade	**8.F.B Use functions to model relationships between quantities.** ○ **8.F.B.4** - Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (*x, y*) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
Algebra 1	**ELG.MA.HS.F.4: Build a function that models a relationship between two quantities.** ○ **F-BF.1** Write a function that describes a relationship between two quantities.★ ○ **F-BF.1.a** Determine an explicit expression, a recursive process, or steps for calculation from a context. Note: Functions may include linear, quadratic, exponential, polynomial (quadratic or cubic), square root, cube root, and piecewise-defined functions (including step and absolute value).
Algebra 2	**Build a function that models a relationship between two quantities.** ○ **F-BF.1** Write a function that describes a relationship between two quantities.★ ○ **F-BF.1.a** Determine an explicit expression, a recursive process, or steps for calculation from a context. ○ **F-BF.1.b** Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model. Note: Functions may include linear, quadratic, exponential, polynomial, square root, cube root, piecewise defined (including step and absolute value), **rational, trigonometric, and logarithmic.**

Students will demonstrate command of the ELG by:

- Interpreting a table to write an appropriate function.
- Applying knowledge of recursive properties of linear, quadratic, and exponential functions to write equations.
- Explaining choice of function type.
- Modeling a relationship between two functions.

Vocabulary:

- constant function
- explicit expression
- exponential growth
- quadratic function
- recursive process

Sample Instructional/Assessment Tasks:

1) **Standard(s):** ELG.MA.HS.F.4 (F-BF.A.1)

Lake Algae (task to be used with Unit 5)

Source: Illustrative Mathematics
https://www.illustrativemathematics.org/content-standards/HSF/BF/A/1/tasks/533)

Item Prompt:

On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

When will the lake be covered half-way?

On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?

On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

Correct Answer:

Since the algae cover doubles every day, if the lake is completely covered on June 30, then one day earlier, on June 29, half of the lake was covered.

On June 29, half the lake was covered. On June 28 one fourth of the lake was covered. On June 27 one eighth of the lake was covered. On June 26 only one sixteenth of the lake was covered by algae.

When the friend does not believe the warning, only $\frac{1}{16}$, or 6.25% of the water surface was covered, so $\frac{15}{16} = 93.75\%$ of the water surface was still open. The algae cover did not seem very dramatic at this point.

After the clean-up crew removes most of the algae, only 1% of the water surface is covered by algae. However, because these algae still grow just as quickly as before, the area covered still doubles every day. Therefore, after one day, 2% will be covered, after two days 4% will be covered, and so on. A table of values shows the following:

n, number of days since clean-up	percentage of lake covered with algae
0	1
1	2
2	2^2=4
3	2^3=8
4	2^4=16
5	2^5=32
6	2^6=64
7	2^7=128

Of course, it is impossible to cover more than 100% of the water surface, so the last entry in the table does not make sense. But we can conclude that sometime between day 6 and 7 the entire lake will be covered by algae. The lake clean-up will be undone completely in less than a week.

Let t be the time (in days) since the algae was introduced into the lake, and p be the percentage of the lake that is covered by algae at time t. We know that June 1 corresponds to $t = 0$ and on June 30 (which corresponds to $t = 29$) that $p = 100$. Since the percentage of the lake that is covered in algae is doubling every day, if p_0 is the percentage of the lake covered when the algae is first introduced, we know that the equation must be of the form:

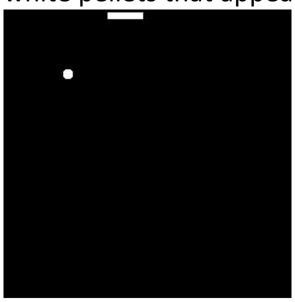
$p(t) = p_0 2^t$.
It remains to find p_0. Since
$p(29) = p_0 2^{29} = 100$,
we conclude that
$p_0 = \frac{100}{2^{29}} \approx 1.86\times10^{-7}$.
So the equation giving the percent covered at time t (for $0 \le t \le 29$) is
$p(t) = (1.86\times10^{-7})2^t$.

2) **Standard(s):** ELG.MA.HS.F.4 (F-BF.A.1)

Snake on a Plane (task to be used with Unit 5)

Source: Illustrative Mathematics
https://www.illustrativemathematics.org/content-standards/HSF/BF/A/1/tasks/533)

Item Prompt:
In a video game called Snake, a player moves a snake through a square region in the plane, trying to eat the white pellets that appear.



If we imagine the playing field as a 32-by-32 grid of pixels, then the snake starts as a 4-by-1 rectangle of pixels, and grows in length as it eats the pellets:

- After the first pellet, it grows in length by one pixel.
- After the second pellet, it further grows in length by two pixels.
- After the third pellet, it further grows in length by three pixels.
- and so on, with the n^{th} pellet increasing its length by n pixels.

Let L(n) denote the length of the snake after eating n pellets. For example, L(3) = 10.

a. How long is the snake after eating 4 pellets? After 5 pellets? After 6 pellets?
b. Find a recursive description of the function L(n).
c. Find a non-recursive expression for L(100), and evaluate that expression to compute L(100).
d. What is the largest number of pellets a snake could eat before he could no longer fit in the playing field? That is, how long is a perfect game of snake?

Correct Answer:

Since L(3) = 10, we can most easily compute L(4) by adding 4 to L(3), to get L(4) = L(3) + 4 = 10 + 4 = 14, rather than starting from scratch, that is by calculating L(4) = 4 + 1 + 2 + 3 + 4 = 14. Similarly, we find L(5) = L(4) + 5 = 14 + 5 = 19, and L(6) = L(5) + 6 = 19 + 6 = 25.

Generalizing the previous part, to compute the length after n pellets, we take the length after n−1 pellets, and add on n more pixels. Algebraically, this reads L(n) = L(n−1) + n.

To complete the specification of a recursive function, we also need to include a starting value, which we are given as L(0) = 4.

The recursive definition gives L(100) = L(99) + 100, but this doesn't particularly help for computing this value. Instead, the "start from scratch" method gives

$$L(100) = 4 + 1 + 2 + 3 + \cdots + 98 + 99 + 100.$$

One particularly neat method for evaluating this proceeds as follows:

1 + 2 + 3 + ⋯ + 98 + 99 + 100 = (1 + 100) + (2 + 99) + (3 + 98) + ⋯ + (50 + 51) = 50·101 = 5050.

We conclude that L(100) = 4 + 5050 = 5054. After 100 pellets, the snake will be 5054 pixels long.

We begin by noting that the 32-by-32 grid of pixels contains 32^2 = 1024 pixels, so by the last part the snake definitely runs out of room by the time it eats 100 pellets. We are looking for the largest number n such that L(n) < 1024, which algebraically takes the form 4 + 1 + 2 + 3 + ⋯ + n < 1024.

Here we can either repeatedly apply the trick of the last part of this problem to solve for n by trial-and-error, or generalize the previous part of the problem to arrive at the formula

$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$:

the method from part (c) applies directly if n is even while if n is odd we find $\frac{n-1}{2}$ groups of n+1 along with the middle number $\frac{n+1}{2}$ and, adding these up, we check that the formula still holds. Substituting this formula into L(n) < 1024, we need to solve $4 + \frac{n(n+1)}{2} < 1024$ or $n(n+1) < 2040$. Again, we have a variety of methods for finding the largest such n, ranging from educated trial-and-error to solving the quadratic equation $n^2 - 2040 = 0$, which gives n ≈ 44.7. With either method, we conclude that the snake can eat 44 pellets before running out of room.